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A Note on Leonardo’s Combinatorial Approach

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Abstract

The purpose of this research is to carry out a study of Leonardo's combinatorial approach so that it is possible to visualize these numbers through combinatorial interpretation. Thus, research is being developed regarding methods and approaches to linear and recurring sequences, based on the combinatorial study of the Fibonacci sequence. In fact, the Fibonacci sPquence is related to other sequences, one of which is the Leonardo sequence, which has similarities with the Fibonacci numbers according to some researchers in the field. Given this scenario, the present research addresses the combinatorial interpretation of Leonardo's sequence, allowing the definition of Leonardo's combinatorial model, considering the notion of board and bracelets in Lucas' sequence. As research results, the study deals with the integration of sequence content with the area of Combinatorial Analysis, allowing a mathematical advancement of Leonardo's sequence. Furthermore, you can visualize the sequence numbers in front of the tiles. The aspects studied in this research are linked to the teaching of sequences in the History of Mathematics, allowing the teaching of Mathematics.

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INTRODUCTION

The Fibonacci sequence was created by mathematician Leonardo Pisano (1180-1250) known as Fibonacci or “son of Bonaccio.” Born in Pisa, Italy, Leonardo acquired mathematical knowledge of the Arab world and in the areas of Algebra and Arithmetic, being remembered because of the problem of reproduction of immortal rabbits, thus generating the Fibonacci sequence (Santos, 2017). Fibonacci wrote 5 works, among which one of the proposed problems stands out, which marks the genesis of the Fibonacci method: “How many pairs of rabbits will be produced in a year, starting with a single pair, if each month each pair generates a new pair that becomes productive from the second month onwards, as long as no rabbit dies during this period?" of the Santos (2017). When answering this question there is a sequence of numbers, a sequence called a sequence of numbers.

Faced with this question, the Fibonacci sequence can be presented, where having the recurrence formula given by $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ and n belongs to the set $\mathbb{N}$ and $F_0 = F_1 = 1$.

This sequence inspired other sequences, highlighting Leonardo's sequence for this research. Leonardo's sequel, has few works referring to these numbers found in the literature, therefore highlighting the works of Catarino and Borges (2019), Shannon (2019) and Vieira et al. (2019), in which they define these numbers as second order. In these researches, the absence in relation to the historical process of these numbers is notable, reporting only its mathematical evolution. In fact, no research has been found confirming the creator of this sequence, however, it is believed that, as it has the name of Leonardo's sequence and great mathematical similarities, it was created by mathematician Leonardo Pisano (1180-1250), the same creator of the Fibonacci sequence. Comparing it with the sequence of Fibonacci, we can see that the value 1 was added at the end of the recurrence $Le_n = Le_{n-1} + Le_{n-2} + 1$, $n \geq 2$ and n belongs to the set $\mathbb{N}$ and $Le_0 = Le_1 = 1$.

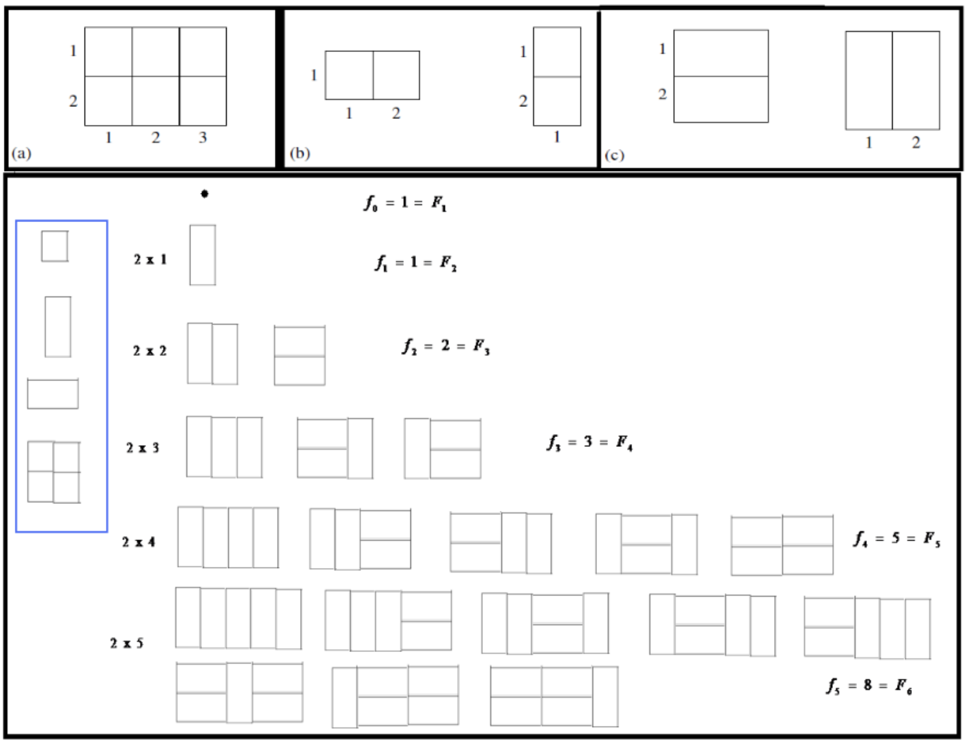
The numerical properties of these numbers are analogous to the Fibonacci properties. Thus, Catarino and Borges (2019) established another recurrence relationship, where from recurrence as being $Le_n = 2\ Le_{n-1} + Le_{n-3}$, $n \geq 3$. The characteristic equation of this sequence can be written based on this recurrence, transforming it into a third-order sequence order. Catarino and Borges (2019) also established a relationship between Leonardo's numbers and Fibonacci, being this $Le_n = 2\ F_n - 1$. So we have $x^3 - 2x + 1 = 0$ presenting three real roots, one being equal to 1 and the other two equal to the roots of the characteristic Fibonacci equation, highlighting the presence of the golden number (value approximately 1.61) as a result of one of the positive roots. Given the brief discussion around the Fibonacci and Leonardo sequences, we have the study of the combinatorial model of Leonardo.

In the context of education, it can be noted that the visualization of the terms in the sequence allows a better understanding of the content, becoming a facilitator of the teaching process. In this way, he was motivated by the study of Leonardo's sequence in light of its combinatorial interpretation.

This passage discusses the historical significance of the Fibonacci Sequence in mathematics and highlights the use of combinatorial interpretations in Elementary Mathematics. It emphasizes the connection between the Fibonacci Sequence and tiling and decomposition of integers. Additionally, it mentions the involvement of prominent researchers in this field (Alves & Sousa, 2023).

Figure 1 deals with a Fibonacci combinatorial approach, using the $2\times n$ board. With this, pieces are available so that you can fill the board and visualize the numbers in the Fibonacci sequence.

Furthermore, the integration between sequences and Combinatorial Analysis can enrich the teaching of other numerical sequences, providing students with a solid foundation to understand and explore different types of sequences and their combinatorial properties. This interdisciplinary approach can be a promising way to promote a more comprehensive and meaningful teaching of numerical sequences, preparing students to face diverse mathematical challenges.

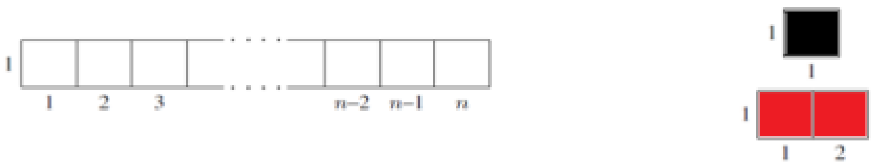


Source: Alves and Sousa (2023)

Figure 1. One interpretation of Fibonacci

FIBONACCI COMBINATORIAL INTERPRETATION

In a generalist way, it is possible to verify contemporary research related to the study of numerical sequences recurring elements and their countless generalizations, which tend to be neglected in books on the History of Mathematics (Grimaldi, 2012; LaGrange, 2013; Stillwell, 2010) An interesting approach is the combinatorial interpretation of the Fibonacci sequence via tiles. So, based on the recurrence of the Fibonacci sequence, given by $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, whose initial values are given by $F_0 = F_1 = 1$. Thus, we have a $1 \times n$ type board, with two types of tiles: one 1×1 black tile and a 1×2 red tile (dominoes). This one-dimensional n–board is indicated in Figure 2.



Source: Benjamin and Quinn (2003a)

Figure 2. Interpretation of the notion of n–board

Definition 2.1: A board is made up of squares called houses, cells or positions. These positions are enumerated and these enumerations describe the position. A given board will just be called an n–board Spreafico (2014).

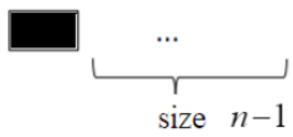
Theorem 2.2: The number of ways to cover an n–board with 1×1 squares and 1×2 dominoes is equal to $f_n = F_{n+1}$ (Spivey, 2019).

Proof. Let S_n be the sum of the tile decks, where $S_0 = 1 = F_1$, $S_1 = 1 = F_2$.

Considering an arbitrary tiling of size n, where $n \geq 2$, we have:

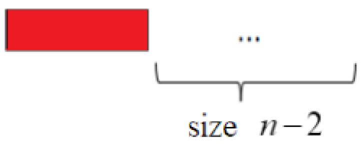
Case 1: Assuming that the tiling starts with the square. These pieces are made up of a square of size 1×1, followed by a screen of size $n-1$ (Figure 3).

The sum is given by S_{n-1}.



Source: Prepared by the authors
Figure 3. Board of size $n-1$

Case 2: Assuming that the tiling starts with a domino of size 1×2. The pieces consist of a board of size $n-2$ and a domino (Figure 4).



Source: Prepared by the authors
Figure 4. Board of size $n-2$

The sum is given by S_{n-2}.

Therefore, according to the principle of addition of the analyzed cases, $S_n = S_{n-1} + S_{n-2}$. Satisfying the Fibonacci recurrence ($F_{n+1} = F_n - F_{n-1}$), with initial conditions $S_0 = F_0$ and $S_1 = F_1$. S_0, $S_n = F_{n+1}$.

In all, the research is based on the works of Benjamin and Quinn (2003b, 2003a), Spreafico (2014) and Koshy (2019), of which describe the notion of an n–board, as well as definitions of terms and expedients to be used in this research, following with the combinatorial properties of the Fibonacci sequence.

Figure 5 shows the cases in the Fibonacci combinatorial model in another format, where 1 represents the square and 2 represents the domino.

Based on the studies carried out on the Fibonacci combinatorial model, we have investigated Leonardo's combinatorics approach. It's worth noting that these connections are often overlooked in specialized History of Mathematics textbooks and are seldom discussed in mathematics teacher training, as mentioned by De-Temple and Webb (2014), Koshy (2019), Spivey (2019) and Vorobiev (2000).

$f_1=1$	$f_2=2$	$f_3=3$	$f_4=5$	$f_5=8$	$f_6=13$		$f_7=21$	
1	11	111	1111	11111	111111	1221	1111111	12112
	2	12	112	1112	11112	1122	111112	21112
		21	121	1121	11121	2112	111121	12121
			211	1211	11211	2121	111211	21211
			22	2111	12111	1212	112111	12112
				122	21111	222	121111	12211
				212	2211		211111	11221
				221			11122	21112
							11212	1222
							2221	2122
								2212
$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$		$n=7$	

Source: Benjamin and Quinn (2003b)

Figure 5. Fibonacci's combinatorial model

A COMBINATORIAL APPROACH TO LEONARDO NUMBERS

In this section, we will follow the discussions highlighted by Koshy (2019) and Benjamin and Quinn (2003b) in which they analyze the combinatorial behavior of the Fibonacci sequence via tiling. Still, we have the relationship established by Catarino and Borges (2019), given by: $Le_n = 2Le_{n-1} - Le_{n-3}$, $n \geq 3$.

With this, we have Leonardo's combinatorial model through tiling and using white squares and dominoes grays and blacks to compose this model. From these available pieces, it is possible to establish a combinatorial interpretation for Leonardo's numbers. The set of objects that are considered is based on tiles of a size range n of squares, that is, a line of n squares.

Let ln be the number of tile shapes on the board of size n and the strip be a board of size n, The following pieces are available: white squares of size 1×1, gray dominoes and black dominoes of size 1×2.

Knowing that gray dominoes and black dominoes cannot occupy the same board, and can only appear alone or with squares.

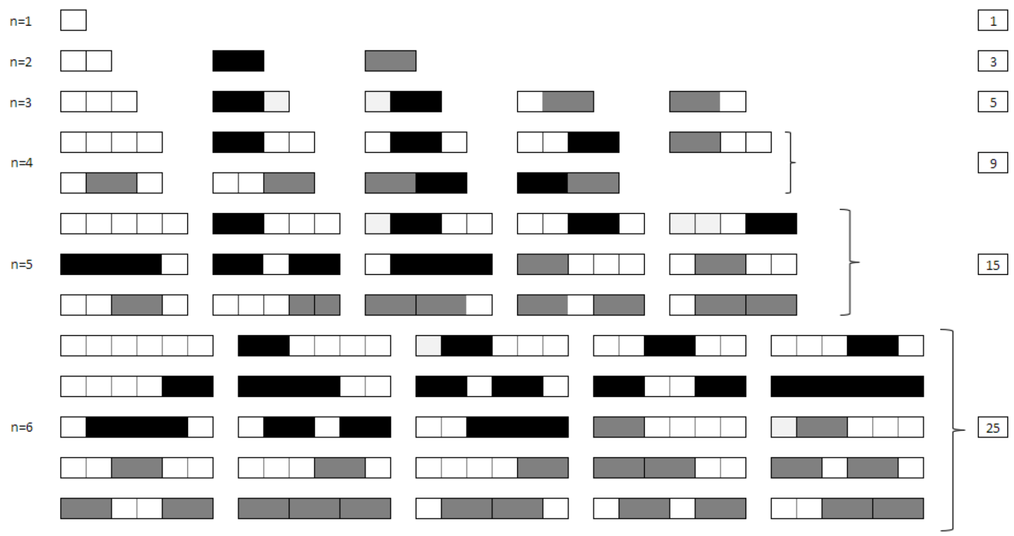
Theorem 3.1: For $n \geq 1$, the possible tiling of a $1\times n$ board, with white squares and gray dominoes and black dominoes, is given by:

$$le_n = Le_n,$$

where le_n is the number of ways to fill the $1\times n$ board and Le_n is the nth term of the Leonardo sequence.

Proof. Carrying out the demonstration by dividing the set len into subsets, we have that: Assuming that $n \geq 3$ and divide into two subsets: Q those that start with squares and D those that start with dominoes (red or blue). Like this $le_n = Q \cup D$ and $|\, le_n \,| = |\, Q \,| + |\, D \,|$. Soon $|\, Q \,| = le_{n-1}$ and $|\, D \,| = le_{n-1} - le_{n-3}$, resulting in $le_n = |\, Q \,| + |\, D \,| = le_{n-1} + le_{n-1} - le_{n-3} = 2\, le_{n-1} - le_{n-3}$. Therefore, le_n satisfies the same recurrence formula with initial conditions of Le_n. Therefore, $le_n = Le_n$, for everyone $n \geq 1$.

For better understanding of the reader, Figure 6 shows Leonardo's combinatorial model for the case $n = 1$, with 1 form of tiling. For the case $n = 2$, there are 3 ways of tiling. For n = 3, there are 5 ways of tiling. For n = 4, we have 9 ways of tiling. For n = 5, there are 15 ways to tile. For n = 6, there are 25 ways to tile.



Source: Prepared by the authors

Figure 6. Leonardo's combinatorial model

CONCLUSION

Due to a broad scenario and interest in research involving forms of generalization of linear sequences and recurring, we have the combinatorial approach of Fibonacci and Lucas sequences as a basic contribution to this research.

On the other hand, the combinatorial approach is presented as an important aspect in research advances involving sequences. In this way, through the notion of board, we have the introduction of the combinatorial study of the sequence of Leonardo, allowing a study of these numbers in the combinatorial scenario of recurring sequences.

In the preceding sections, we explored basic problems in Combinatorics. Surprisingly, these problems led to connections with the Fibonacci Sequence. In summary, the problems and approaches discussed in the previous sections belong to the field of Pure Mathematics known as "Combinatorics," often referred to as Finite Mathematics because it focuses on studying finite objects. However, it's essential to note that there are infinitely many finite objects, and at times, it's beneficial to reason about all members of an infinite collection.

For the area of education, it is noted that by visualizing the terms of Leonardo's sequence, it is possible to advance the mathematical content and improve the Mathematics teaching process for the History of Mathematics area. Thus, the teacher will have a new possibility of approaching the content in the classroom, allowing a didactic transposition with the combinatorial interpretation of the sequence under study.

Given the presentation of the combinatorial approach of Leonardo's sequence, it can be noted that the objective of the work was achieved, carrying out a study of these numbers and allowing an improvement in the Mathematics teaching process.

This perspective on Combinatorics not only enhances the understanding of mathematics for teachers from a basic level but also illustrates its relevance in contemporary research related to Fibonacci numbers and Leonardo numbers.

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