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Some Elementary Combinatory Properties and Fibonacci Numbers

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Abstract

In general, in the midst of History of Mathematics textbooks, we are faced with a discussion due to curiosity about the emblematic Fibonacci Sequence, whose popularization occurred with the proposition of the reproduction model of immortal rabbits. On the other hand, in the comparison of the multiple approaches and discussions of certain subjects in Elementary Mathematics, in the present work, we highlight combinatorial interpretations that, with the support of a characteristic and fundamental reasoning for the mathematics teacher, can be generalized and formalize some eminently intuitive components. In particular, this work deals with properties derived from the notion of tiling and decomposition of an integer that, depending on the board, will correspond to the numbers of the Fibonacci Sequence. We bring a theoretical discussion supported by great names that research in this area.

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INTRODUCTION

Significant traces of Combinatorics principles can be found in several civilizations and remote times. In this primitive scenario, the authors Wilson and Watkins (2013) indicate multiple meanings, such as: energetic, poetic, mystical, educational, etc. We have an example in Figure 1, where the authors discuss manuscripts by Ramon Llull, Duke of Venice, in 1210.

In this example, a chapter of the work *Ars Compendiosa Inveniendi Veritatem* (The Concise Art of Finding the Truth, English translation) began by listing sixteen attributes of God: goodness, greatness, eternity, power, wisdom, love, virtue, truth, glory, perfection, justice, generosity, mercy, humility, sovereignty, and patience. So, Ramon Llull wrote, combinatorially $C_{16,2} = 120$ short essays of about 80 words each, considering God's goodness related to greatness (Wilson & Watkins, 2013).

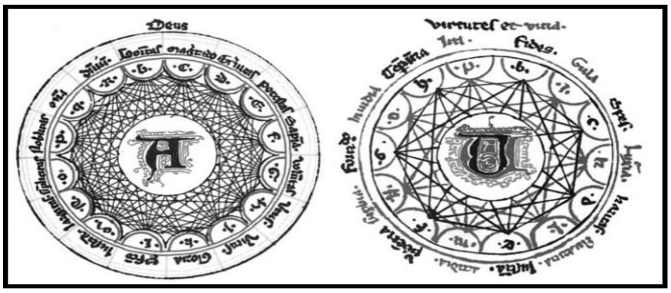

Figure 1. Wilson and Watkins (2013) recover primitive concepts that gave rise to Combinatorics.

As a contemporary motivation on the use of Combinatorics, and for a preliminary discussion, let us consider particular decompositions of the positive integer '7' Indeed, trivially, we have: $\{1+1+1+1+1+1+1; 1+2+2+2; 1+3+3;$ $1+2+4; 1+6; 3+1+3; 7; \ldots\}$. In the previous set we can see some examples called *compositions of the number 7*. On the other hand, when we examine the order of the terms, we can write $7=1+6$ or $7=6+1$, which represent different compositions for the positive integer $n=7$. Furthermore, compositions of the particular type or species may occur $7=3+1+3$, which represents the same composition, in any of the senses that we consider. When this occurs, we say we have a palindrome (Grimaldi, 2012). Let's see another example in more detail.

In fact, we can say that there are 16 ways to write the positive integer $n=5$ as the sum of integers (see Figure 2). Note that when order is not relevant, compositions $5=4+1=1+4$ are considered equal. In any case, following the reasoning employed by Grimaldi (2012), we seek to determine a formula for the number of compositions of a positive integer $n \in \mathrm{N}$. In a heuristic way, the author advises to consider a trivial composition for the integer $n=5=1+1+1+1+1$, counting the number of installments (5) and the number of additive operations (4 times) (Figure 2):

(1) 5	(5) $2+3$	(9) $2+2+1$	(13) $1+2+1+1$
(2) $4+1$	(6) $3+1+1$	(10) $2+1+2$	(14) $1+1+2+1$
(3) $1+4$	(7) $1+3+1$	(11) $1+2+2$	(15) $1+1+1+2$
(4) $3+2$	(8) $1+1+3$	(12) $2+1+1+1$	(16) $1+1+1+1+1$

Figure 2. Grilmaldi (2012) discuss the compositions of the positive integer $n=5$.

Intuitively, and when comparing with the data in Figure 2, for the set $\{1, 2, 3, 4\}$ we could determine that $2^4 = 16$ represents the number of subsets of all partitions. On the other hand, if we take a subset $\{1, 3\} \subset \{1, 2, 3, 4\}$, we can form the following compositions: $(1 \underset{1^a}{+} 1) + (1 \underset{3^a}{+} 1) + 1$ and $(1 \underset{1^a}{+} 1) + (1 \underset{3^a}{+} 1 + 1)$ considering the position of additive operations '+' in 1st and 3rd position in the decomposition.

Grimaldi (2012) observes that the subset indicates that parentheses should be placed around the '1' in the 1st and 3rd positions, where addition operations occur.

Still on Figure 2, we identify the subset of compositions that involve only the digits '1' and '2' in the composition of the positive integer $n=5$. Note that if we eliminate all compositions (8 compositions), except those with the digits '1' and '2', we can determine $2^4-8=16-8=8=F_6$ (later, Table 1). We can see (Figure 2) that only two palindromes occur:

$$2+\underbrace{1}_{\text{central term}}+2,\quad 1+1+\underbrace{1}_{\text{central term}}+1+1.$$

We can easily observe the existence of central terms in both palindromes. In Figure 3 we can see the example of a palindrome, with an odd central term, within the case of integer compositions $n=5$.

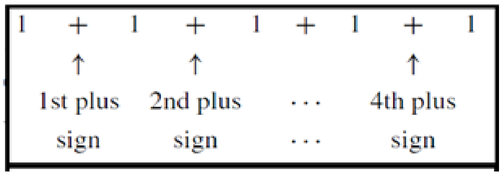

Figure 3. Grilmaldi (2012) discuss the compositions of the positive integer $n=5$ considering the operations.

Before proceeding to the subsequent sections, it is essential to point out the considerations of De-Temple and Webb (2014), when they examine some standard procedures in solving combinatorial problems, namely: (i) introduce a notation, with respect to which h_n represents an answer to be determined in the n-th case; (ii) determine some particular initial values h_1, h_2, h_3, h_4, etc. by direct count; (iii) employ combinatorial reasoning in order to determine the recurrence relation that expresses h_n depending on the previous values of the sequence; (iv) solve the recurrence relation in order to find a unique solution.

In this work, we seek to carry out a bibliographical survey that supports a mathematical discussion about some combinatorial problems that have varied interpretations, based on works of great names within the research in this field of knowledge such as Benjamin and Quinn (1999; 2003), Hemenway (2005), Koshy (2001; 2019), Grimaldi (2012), Singh (1985) and Vajda (1989). His works relate to the emblematic Fibonacci Sequence and its development. Such an approach is usually neglected by History of Mathematics textbooks, which place too much emphasis only on its anecdotal aspects and a bias that does not involve a content that overcomes curiosity for the history of the production of the 'immortal rabbits' (Figure 4).

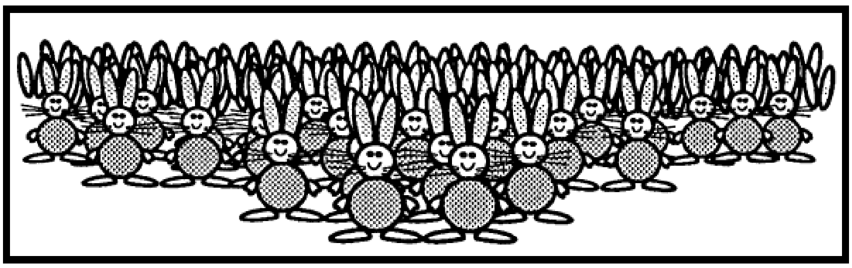

Figure 4. Gullberg (1997) discusses approaches to the Fibonacci Sequence.

Given the above, we seek to discuss the following question: "What elementary properties of Combinatorics allow a meaning and/or interpretation for the set of numbers that occur in the Fibonacci Sequence?" Thus, in the following sections we bring a theoretical discussion based on a mathematical approach to the subject.

COMPOSITIONS, PALINDROMES AND FIBONACCI NUMBERS

In the introductory section we found, in a heuristic way, that there is a correspondence involving the set of compositions of the positive integer $n=5$ and the subsets of the set $\{1,2,3,4\}$, which correspond to the arithmetic expression $2^4=16$ and that represents the subset quantity of all your partitions. Inductively, according to Grimaldi (2012, p. 25), we could write that, "given a positive integer $n \in \mathrm{N}$, we can determine the number 2^{n-1} as the total of compositions".

However, how do the previous properties relate, more precisely, to the Fibonacci sequence defined by the relation $F_{n+1}=F_n+F_{n-1}$, with initial values $F_0=0$ and $F_1=1$?

About the Fibonacci Sequence and Some Elementary Properties

For the purpose of the problem that we seek to discuss, let us consider the recurrence $F_{n+1}=F_n+F_{n-1}$ and the initial values $F_0=0$ and $F_1=1$, that determine the values in Table 1.

Table 1. Description from the recurrence of the Fibonacci numbers, for $F_n, n \geq 1$

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	**...**	F_n
0	1	1	2	3	5	8	13	21	34	55	89	144	233	...	...

Strictly speaking, let's consider compositions of a positive integer $n \in \mathrm{N}$, considering only when only the digits '1' and '2' occur. To exemplify, let's consider Figure 5, suggested by Grimaldi (2012). Indeed, we determine some compositions of the positive integers $n=3,4,5,$ as we can see:

$$(n=3): 2+1, 1+2, 1+1+1$$
$$(n=4): 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1$$
$$(n=5): 2+2+1, 2+1+1+1, 1+2+1+1, 1+1+2+1,$$
$$1+1+1+1+1,$$
$$2+1+2, 1+2+2, 1+1+1+2$$

Figure 5. Grimaldi (2012) discuss compositions of the integers $n=3,4,5$ with only digits '1' and '2'.

Grimaldi (2012) defines the term c_n which it interprets as the number of compositions, when the digits '1' and '2' occur. From an arithmetic point of view and with the support of Figure 5, we can determine, as an arithmetic sum, that $c_5=8=F_6=5+3=c_4+c_3$. Grimaldi (2012, p. 25) explains that "the first five compositions of $n=5$ can be obtained by adding the term '+1', from the five previous compositions of $n=4$.

The previous argument supports the definition of recurrence $c_n = c_{n-1} + c_{n-2}$, $n \geq 3$, where we can easily understand that $c_1 = 1$ and $c_2 = 2$ correspond to the number of compositions of the integers $n = 1, 2$, $c_1 = 1$ and $c_2 = 2$, respectively. Furthermore, without further details, Grimaldi (2012) assumes that $c_n = F_n$, $n \geq 1$. It should be noted that in the previous examples we can determine the number of palindromes present in each decomposition. Indeed, if we seek to determine the compositions of the integer $n = 11 \therefore c_{11} = F_{12} = 144$, for example. In this case, for the odd integer $n = 11$, when we consider their decompositions in terms of '1' and '2', each palindrome must contain an odd central term. For example, when we write:

$$1+1+1+1+1+\underbrace{1}_{\text{central term}}+1+1+1+1+1,$$

because the digit '1' is the only possibility for a central term in the palindrome. Grimaldi (2012) observes that, for this palindrome we can consider, the set of compositions of the integer $n=5$, that correspond to the value $n = 5 \therefore c_5 = F_{5+1} = F_6 = F_{\frac{11+1}{2}} = 8$. Furthermore, for this set of 8 compositions, with the central term fixed at '1', we determined all the palindromes present in the decomposition of the integer, when considering the digits '1' and '2'.

From this particular case, Grimaldi (2012, p. 26) states that "in general, if the positive integer n is odd, when we consider the set of compositions $c_n = F_{n+1}$, we determine that $F_{\frac{n+1}{2}}$ correspond precisely to the set of palindromes".

On the other hand, when dealing with an even integer we have, for example, $n = 12 \therefore c_{12} = F_{13} = 144$. Let's take the palindrome:

$$[2+2+2]\underbrace{+}_{\text{central term}}[2+2+2]$$

On the right side, we see a composite of the integer $n = 6 = 2+2+2$ for which the following correspondence holds $n = 6 \therefore c_6 = F_{6+1} = F_7 = 13$. However, compositions (palindromes) of the form:

$$1+1+1+1+1+\underbrace{2}_{\text{central term}}+1+1+1+1+1$$

for the positive integer $n=12$, whose central term is an even number and, necessarily, the central term could not be odd, whose number of compositions are determined by $c_{12} = F_{12+1} = F_{13} = 233$.

Before finishing this section, we once again refer to Grimaldi (2012), which establishes a way to determine the number of palindromes in a composition. Indeed, considering the positive integer $n = 12$, to determine the palindromes in the set of compositions $F_{13} = 233$, we consider the cases: (i) if the central term of a composition is a plus sign '+' we consider it in the form of a 'reflection in the mirror'; examining the compositions of the integer $n = 6$, such as in the case of

$$\underbrace{[2+2+2]+[2+2+2]}_{\text{central term}}$$

or

$$\underbrace{[1+1+1+1+1+1]+[1+1+1+1+1+1]}_{\text{central term}}$$

and which are equivalent to the amount of $n = 6 \therefore F_7 = 13$ palindromes; (ii) If a number occurs as the central term, Grimaldi (2012) states that it must be even (that is, the digit '2' must occur), as in the previous example we wrote:

$$\underbrace{[1+1+1+1+1]+2+[1+1+1+1+1]}_{\text{central term}}$$

and one must consider, in this case, the compositions of the integer $n = 5 \therefore c_5 = F_{5+1} = F_6 = 8$ palindromes.

Finally, by an additive principle, when considering the whole set of palindromes, we write $F_6 + F_7 = F_8 = 21$ palindromes present in the composition of the positive integer $n = 12$.

WAYS TO COMPLETE A BOARD AND SOME THEOREMS

In the previous section we pointed out some properties that, through combinatorial arguments, reveal properties intrinsically related to the Fibonacci Sequence. Preserving some of the previous arguments, we have Table 2, which allows relating the number of compositions of an integer n:

Table 2. Compositions c_n for a positive integer n from the digits ‘1’ and ‘2’.

$f_1 = 1$	$f_2 = 2$	$f_3 = 3$	$f_4 = 5$	$f_5 = 8$	$f_6 = 13$		$f_7 = 21$	
1	11	111	1111	11111	111111	1221	1111111	12112
	2	12	112	1112	11112	1122	111112	21112
		21	121	1121	11121	2112	111121	12121
			211	1211	11211	2121	111211	21211
			22	2111	12111	1212	112111	12112
				122	21111	222	121111	12211
				212	2211		211111	11221
				221			11122	21112
							11212	1222
							2221	2122
								2212
$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$		$n = 7$	

In the interest of providing greater rigor and increasing details in our discussion, we have established, from now on, the following definition:

Definition 1: (Board) The board is a formation of squares called squares, cells or positions. These positions are enumerated and this enumeration describes the position. Such a board will just be called *n*−board (Spreafico, 2014).

Next, we visualize the example of a 4–board. For its filling and possible configurations, the authors Benjamin and Quinn (2003) use only squares 1×1 (in the lighter color) and dominoes 1×2 (in the darkest color). Easily, if we wish to determine all possible tiling sets, including squares and dominoes, with the support of Figure 6 we can conclude that they are a total of $F_5 = 5$ possibilities (Figure 6, left). We could still choose just the set of tiles that have at least one domino (in the darkest color) 1×2 and, in this case, determine compositions of squares and dominoes. That is, we eliminate the first configuration only with the presence of squares 1×1. In Figure 7 we visualize the generalization of an *n*–board and then establish a relationship, in view of Theorem 1, with the Fibonacci sequence:

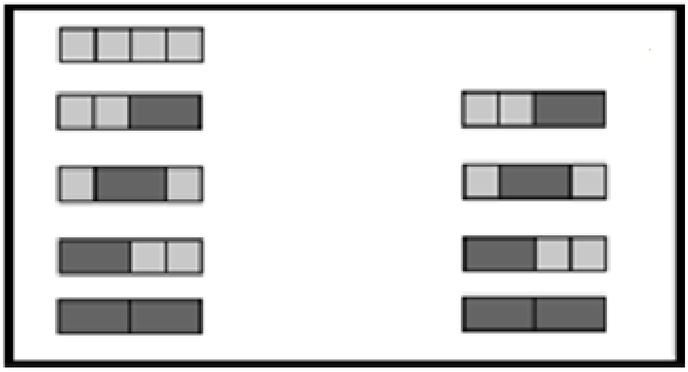

Figure 6. Benjamin and Quinn (2003) provide a particular representation via 4–Board.

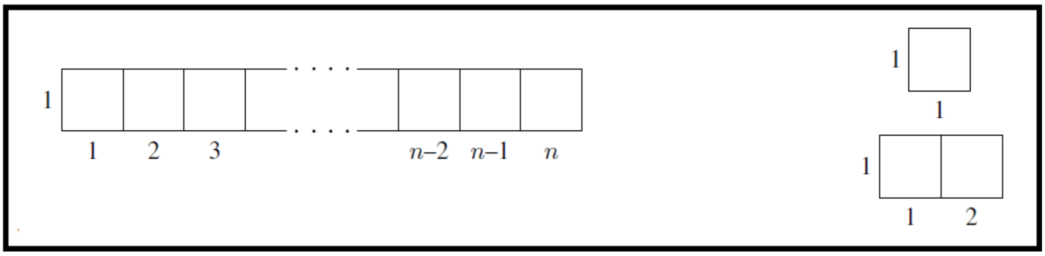

Figure 7. Benjamin and Quinn (2003) provide a representation via 'Board' related to the Fibonacci sequence.

Before proceeding, it is urgent to formalize certain heuristic and intuitive operations and arguments used just now, considering Theorem 1.

Theorem 1: The number of ways to cover a board $1\times n$ with squares 1×1 and dominoes 1×2 is equal to f_{n+1} (Spivey, 2019).

Demonstration: We can define as c_n the number of forms to cover a board $1\times n$ with squares 1×1 and dominoes 1×2. For a board of the type 1×1 we use only a square 1×1, i.e., we have $c_1 = 1 = F_2 = F_{1+1}$. For a board of the type 1×2, we have two possible configurations: with two squares or with one domino 1×2. In this case, note that $c_2 = 2 = F_3 = F_{2+1}$. Then, when we consider the number c_n as the number of ways to cover a board $1\times n$, there are only two possibilities, namely: (i) a set of partitions whose first piece is just a square 1×1; (ii) a set of partitions whose first piece is exactly a rectangle 1×2. if it occurs (i), that is, the first piece is a square, so the other positions $(n-1)$ remaining must correspond to $c_{n-1} = F_n$.

With the same reasoning, if (ii) occurs, that is, the other positions $(n-2)$ remaining must correspond to $c_{n-2} = F_{n-1}$. To consider the set c_n as the total number of ways to cover a board $1\times n$, by an additive principle, we will add $c_n = (c_{n-1} + c_{n-2}) = F_n + F_{n-1} = F_{n+1}$, $n \geq 1$.

In Figure 8, Grimaldi (2012) seeks to determine the amount of filling a board 2×3. The author notes that horizontal and vertical dominoes can be used (see Figure 7, right). The author explains that if we have a board 2×2 we have two ways of tiling: using two horizontal dominoes 1×2 or two vertical dominoes 2×1. If we consider a board 2×3 (Figure 8a), the author suggests decomposing the previous figure and counting the tiles for the case of the board 2×2, in which we have $q_2 = 2$ (Figure 8c).

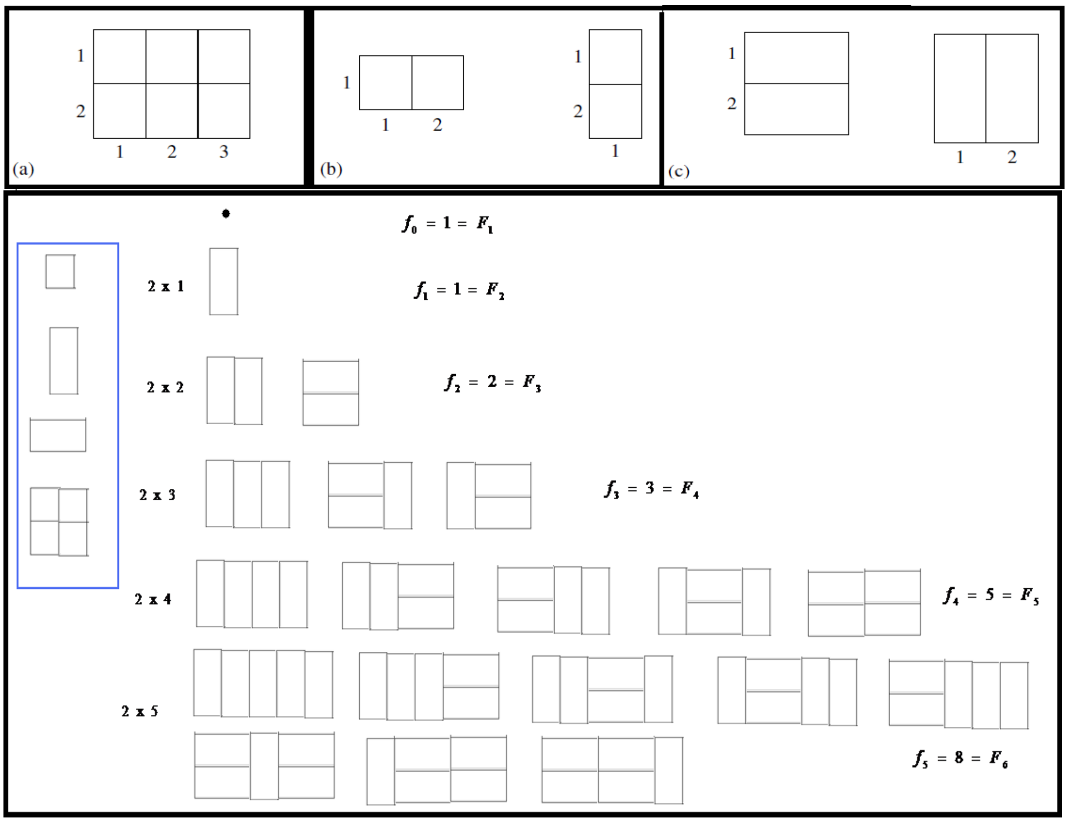

Figure 8. Grimaldi (2012) presents a board 2×3 and describes a relation with the sequence.

Theorem 2: The number of ways q_n to cover a board $2\times n$ with squares 1×1 and dominoes 1×2 is equal to $q_n = F_{n+1}$.

Demonstration: In the general case, for a board $2\times n$, we have two possibilities: (i) the first domino is vertical, so the remaining board will be of the type $2\times(n-1)$ and we will cover it of q_{n-1} distinct ways; (ii) by two horizontally juxtaposed squares, then the remaining board will be of the type $2\times(n-2)$ and we'll cover it for tiling for a total of q_{n-2} distinct ways. Considering both possibilities, by a combinatorial principle, Grimaldi (2012) establishes that $F_n = q_n = q_{n-1} + q_{n-2},\ n\geq 3$ and $q_1 = 1 = F_2,\ q_2 = 2 = F_3$.

COMBINATORY INTERPRETATION OF ELEMENTARY IDENTITIES

In the preceding sections, we used arguments and reasoning eminently of a combinatorial nature, aiming to show a character of arithmetic invariance of the recurrent sequence defined by $F_{n+1} = F_n + F_{n-1}$, with the initial values $F_0 = 0$ and $F_1 = 1$, whose combinatorial properties are usually neglected in History of Mathematics textbooks. Now, let us remember some identities that we found in the literature of this area of knowledge, such as:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

which, according to Koshy (2001) was demonstrated by the French mathematician François Édouard Anatole Lucas in 1876, and similarly, the finite sums

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n}$$

and

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1}\ .$$

Grimaldi (2012) comments that other mathematicians, such as Giovanni Domenico Cassini (1625-1712) and Robert Simson (1687–1768) also found several ways to verify such combinatorial identities. The author also explains that in 1901, Eugen Neto (1846–1919) studied the set of compositions of a positive integer n, with a different method, except for the occurrence of the digit '1', as we can see in Table 3. The author suggests observing arithmetic relations and that, for these initial cases, we can establish the relation $e_n = F_{n-1}, n\geq 1$. On the other hand, Grimaldi (2012) adds that the determination of compositions e_n can be calculated from the recurrence $e_n = e_{n-1} + e_{n-2}, n\geq 3$, with $e_1 = 0, e_2 = 1$.

Table 3. Compositions of a positive integer, except for the digit '1'.

n	e_n	Compositions	n	e_n	Compositions
1	0	-	6	5	6, 2+4, 3+3, 4+2, 2+2+2
2	1	2	7	8	7, 2+5, 3+4, 4+3, 5+2, 2+3+2, 3+2+2, 2+2+3
3	1	3	8	13	8, 2+2+2+2, 2+2+4, 2+4+2, 4+4, 3+5, 5+3, 2+3+3, 2+3+2, 3+2+2
4	2	2+2, 4	9	21	3+3+3, 3+6, 6+3, 2+2+5, 2+5+2, 5+2+2, 4+5, 5+4, 2+2+5, 2+5+2, 5+2+2, 2+7, 7+2,
5	3	5, 2+3, 3+2	10	34	10, 2+2+2+2+2, 4+2+2+2, 2+4+2+2, 2+2+4+2, 2+2+2+4, 4+3+3, 3+4+3, 3+3+4, 6+4, 4+6, 2+8, 8+2, 4+4+2, 4+2+4, 2+4+4, 3+5+2, 3+2+5, 2+3+5, 2+2+7, 2+7+2, 7+2+2, 3+7, 7+3, 5+5, 2+3+2+3, 3+2+2+3, 2+3+2+3, 2+6+2, 6+2+2, 2+2+6, 2+5+3, 5+2+3.

Now, let's return to the set of palindromes that occur in the compositions of the integer n (excepting the digit '1'). When we consider the method proposed by Eugen Neto (1846–1919), we determine the number of compositions $e_n = F_{n-1}, n \geq 1$. To exemplify, let's take $n = 15$. Grimaldi (2012, p. 28) observes that, in the case where n is odd, "then the central term of the sum will always be an odd number". For example, taking the particular composition:

$$2 + 2 + 2 + \underbrace{3}_{\text{central term}} + 2 + 2 + 2$$

The smallest central term to be used will be '3'. In this case, we consider the compositions corresponding to each side of the central term. For example, in the decomposition:

$$[2 + 2 + 2] + \underbrace{3}_{\text{central term}} + [2 + 2 + 2]$$

We consider $[2+2+2]$ and, taking $F_{6-1} = F_5 = 5 = e_6$ that correspond to the integer $n = 6$ (same to the left side). Therefore, there are $F_5 = 5$ palindromes in the compositions of $n = 15$, as central term equal to the digit '3'. If the middle term is

$$[2 + 3] + \underbrace{5}_{\text{central term}} + [2 + 3],$$

We determine that $F_{5-1} = F_4 = 3 = e_5$ there are three compositions of the palindromic type, which are:

$$\underbrace{[3+2]+\ 5\ +[3+2]}_{\text{central term}},\ \underbrace{[2+3]+\ 5\ +[2+3]}_{\text{central term}},\ \underbrace{[5]+\ 5\ +[5]}_{\text{central term}}$$

Consequently, the total amount of palindromes present in the integer decompositions $n=15$ (Table 4) occur, according to the above conditions and using the identity:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 : F_5 + F_4 + F_3 + F_2 + F_1 + F_0 + 1 = \sum_{i=0}^{5} F_i + 1 = (F_7 - 1) + 1 = F_7 .$$

Before wrapping up this section, we examine a combinatorial interpretation for the identity $\sum_{i=1}^{n} F_i = F_{n+2} - 1$, supported by the arguments recorded by Benjamin and Quinn (2003). By Theorem 1, we know that the number of ways to cover a board $1\times n$ with squares 1×1 and dominoes 1×2 is equal to $c_n = F_{n+1}, n \geq 1$.

Table 4. Determination of palindromes by the Eugen Neto method.

Central term	Number of palindromes	Central term	Number of palindromes
3	$F_5 = 5 = e_6$	11	$F_1 = 1 = e_2$
5	$F_4 = 3 = e_5$	13	$F_0 = 0 = e_1$
7	$F_3 = 2 = e_4$	15	1
9	$F_2 = 1 = e_3$		

To verify the identity, we tried to answer the following question: How many tilings of a $(n+2)-\text{board}$ have at least one domino 1×2?

In Figure 8 we can immediately see that for a $(n+2)-\text{board}$, there are $f_{n+2} = F_{n+1}$ possible tilings, however a tiling will not contain any dominoes 1×2 (which in the figure is indicated in gray color). This is the case of tiling only with squares 1×1. It follows from this fact that the total quantity containing at least one domino will correspond to the number $f_{n+2} - 1 = F_{n+3} - 1$ and we exclude the possibility with squares only, similarly to what we did in Figure 6.

Now, by examining Figure 8, we begin to consider the existence of dominoes and the position of the respective domino 1×2. Preliminary, it may occur: i) The position on the domino in $(n+1, n+2)$ and, in this case, only $f_n = F_{n+1}$ tiling in a $n-\text{board}$ (having fixed the position $(n+1, n+2)$); ii) In the next step, the position on the domino can occur in $(n, n+1)$, in which there will be a square in the position $(n+2)$ and in this case there are only $f_{n-1} = F_n$ tiling in a $(n-1)-\text{board}$; (iii) Then the position on the domino can occur in $(n-1, n)$, in which there will be a square in the positions $(n+1)$ and $(n+2)$ and, in this case, only $f_{n-2} = F_{n-1}$ tilings in a $(n-2)-\text{board}$.

Continuing with the previous steps, step by step, we can identify in Figure 9 and observe that the last domino will be in the position $(1, 2)$. So, there must be $f_0 = F_1$ tilings, as there will be a first piece (the domino) and all the rest are made up of squares 1×1:

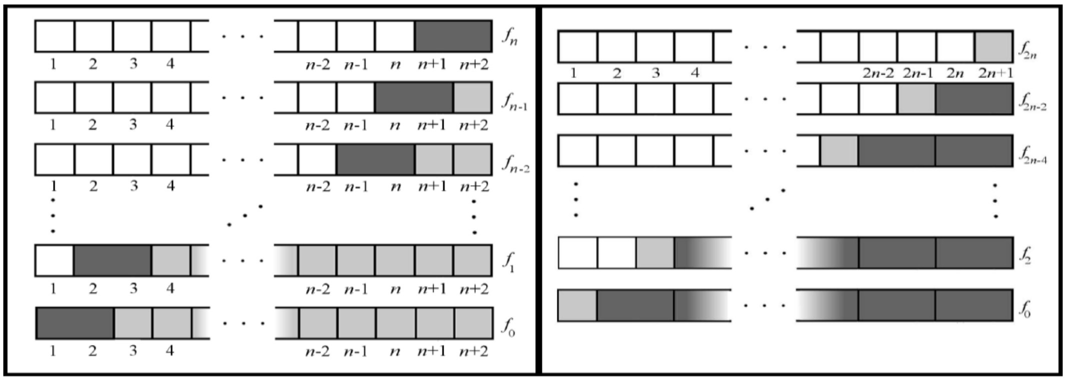

Figure 9. Representation via tiling for identities $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ and $\sum_{i=1}^{n+1} F_i = F_{n+3} - 1$

Finally, summing all contributions involving tiling in steps F_{n+1}, F_n, F_{n-1}, F_{n-2}, ..., F_3, F_2, F_1, 0, it follows that:

$$F_{n+1} + F_n + F_{n-1} + F_{n-2} + \cdots + F_3 + F_2 + F_1 + F_0 = \left(\underbrace{f_{n+2} - 1}_{\substack{N^{\circ} \text{ of the tilings on the (n+2)-board} \\ \text{excluding one tilings with the squares}}} \right) = \left(F_{n+3} - 1\right)$$

Note that the expressions $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ and $\sum_{i=1}^{n+1} F_i = F_{n+3} - 1$ are exactly the same, except for the addition of one term. Still supported by Figure 7, Benjamin and Quinn (2003) provide an interpretation for the identity $\sum_{i=1}^{n} f_{2i} = f_{2n+1}$. For that, we consider a $(2n+1)-\text{board}$. Immediately, as there are an odd number of positions, therefore, every tiling must have at least one square (1×1). In Figure 7 (on the right), the authors Benjamin and Quinn (2003a; 2003b) consider the position of this square, which remains present in all tiling possibilities, since the length of the board is odd. Systematically, it should occur that: i) If the square 1×1 will be in the position $2n+1$ they must occur f_{2n} coverings or tiling forms in the $2n-\text{board}$ remaining; ii) If the square 1×1 will be in the position $2n-1$ they must occur f_{2n-2} coverings or tiling forms in the $(2n-2)-\text{board}$ remaining.

Repeating the previous arguments, we consider all contributions:

$$f_{2n} + f_{2n-2} + f_{2n-4} + \cdots + f_6 + f_4 + f_2 + f_0 = \left(\underbrace{f_{2n+1}}_{N^{\circ} \text{ of the tilings in the (2n+1)-board}} \right)$$

In the previous sections we approached some elementary problems in combinatorics, whose arguments and representations used culminate, surprisingly, with the emergence of relations with the Fibonacci Sequence that, in certain specialized textbooks of History of Mathematics, tend to be neglected.

The topic discussed in this article has the potential to instigate future research on the applications of these sequences in combinatorial number theory and, possibly, other areas involving matrix algebra. In addition, we aim to broaden the study of other algebraic properties of the Fibonacci sequence in general, as well as the possibilities of applications of these sequences in teaching sessions, focused on the initial training of mathematics teachers.

FINAL CONSIDERATIONS

In the previous sections we addressed some elementary problems in Combinatorics whose arguments and representations used culminate, surprisingly, with the emergence of relations with the Fibonacci Sequence which, in certain specialized textbooks of the History of Mathematics are often not addressed and are rarely discussed in mathematics teacher training (De-Temple & Webb, 2014; Koshy, 2019; Spivey, 2019; Vorobiev, 2000).

In particular, the way to consider certain compositions of a positive integer n, with or without the presence of the digits '1' and '2' and, for example, we found that palindromes make it possible to relate and determine sets and subsets that, from a numerical point of view, correspond precisely to the numerical values that we indicate, in addition to the theorems involving tiling with squares and dominoes.

In our works, Alves (2017; 2022) we have indicated a non-static and evolutionary understanding of mathematical knowledge, from the birth stage of more primitive ideas, culminating in a specialized scenario, in which researchers and mathematicians from different countries express an interest in same math problem. In this way, elementary identities like $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ and $\sum_{i=1}^{n+1} f_{2i} = f_{2n+1}$ that held the interest of professional mathematicians in the past (Stillwell, 2010) can be revisited, through combinatorial arguments and expressing heuristic properties for numerous compositions involving Fibonacci numbers (Hoggatt Jr & Lind, 1969).

Finally, the problems and approach we discussed in the preceding sections fall under what in Pure Mathematics we call “Combinatorics, which is often called Finite Mathematics, because it studies finite objects. But there are infinitely many finite objects, and it is sometimes convenient to reason about all the members of an infinite collection.” (Stillwell, 2010, p. 554). In these terms, Combinatorics stimulates the Mathematics teacher's understanding, from an elementary stage to the realization of a modern research scenario that confirms the vigor of research around Fibonacci numbers.

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